AKADEMIA EDUKACYJNO - INŻYNIERSKA

www.e-MECHANiK.com.pl

MATURA STUDIA PRAKTYKA STAŻ PRACA BIZNES PRZEMYSŁ STEAM

KOMPLEKSOWE WSPARCIE EDUKACYJNE **ON – LINE** NA KAŻDYM ETAPIE KSZTAŁCENIA INŻYNIERSKIEGO

email: **kontakt@e-mechanik.com.pl**
web: **www.e-MECHANiK.com.pl**
fb: www.facebook.com/groups/emechanik
tel: **(+48) - 697-165-075**

Wszystkie prawa zastrzeżone All rights reserved

ANALIZA MATEMATYCZNA II

"CAŁKA III – PARAMETRY GEOMETRYCZNE I FIZYCZNE BRYŁ"

PROJEKT

Wyprowadź formułę ogólną na:

- objętość
- masowy moment bezwładności

bryły geometrycznej w postaci jednorodnej kuli.

DANE:

$z(x,y) = OK$

SZUKAM:

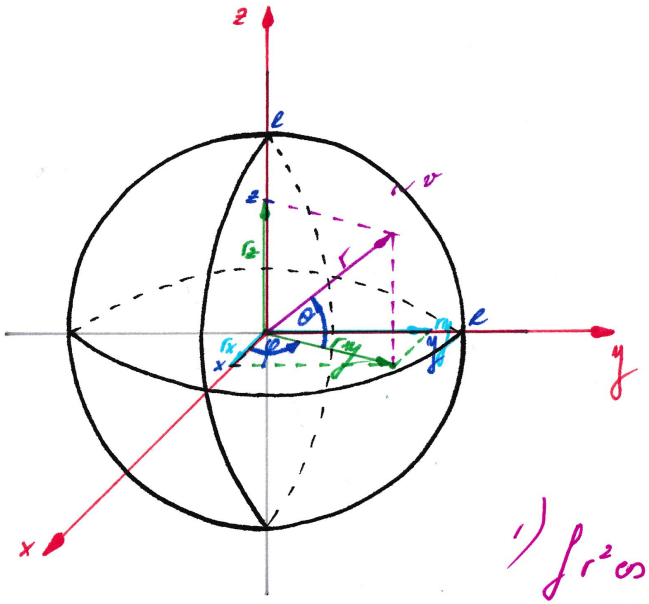
$V = ?$

AM II

$$x^2 + y^2 + z^2 = \ell^2$$

$$z(x,y) = \pm\sqrt{\ell^2 - x^2 - y^2}$$

$$\vartheta = \iiint_V 1\,dx\,dy\,dz = \iiint_V r^2\cos\theta\,d\theta\,d\varphi\,dr$$



$$\frac{z}{r} = \sin\theta \Rightarrow z = r\sin\theta$$

$$\frac{r_{xy}}{r} = \cos\theta \Rightarrow r_{xy} = r\cos\theta$$

$$\frac{x}{r_{xy}} = \cos\varphi \Rightarrow x = r\cos\theta\cos\varphi$$

$$\frac{y}{r_{xy}} = \sin\varphi \Rightarrow y = r\cos\theta\sin\varphi$$

$$\Rightarrow \begin{cases} I)\ x = r\cos\theta\cos\varphi \\ II)\ y = r\cos\theta\sin\varphi \\ III)\ z = r\sin\theta \end{cases}$$

$$IV)\ J = r^2\cos\theta$$

$$\vartheta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\pi} d\varphi \int_0^{\ell} r^2\cos\theta\,dr$$

$$V_{r\varphi\theta}: \begin{cases} 0 \le r \le \ell \\ 0 \le \varphi \le 2\pi \\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{cases}$$

$$\vartheta = \frac{4\pi}{3}\ell^3$$

$$I)\ \int r^2\cos\theta\,dr = \cos\theta\int r^2\,dr = \cos\theta\left[\frac{1}{3}r^3 + C\right]_0^{\ell} = \frac{1}{3}\ell^3\cos\theta$$

$$II)\ \int \frac{1}{3}\ell^3\cos\theta\,d\varphi = \frac{1}{3}\ell^3\cos\theta\int d\varphi = \frac{1}{3}\ell^3\cos\theta\left[\varphi + C\right]_0^{2\pi} = \frac{2\pi}{3}\ell^3\cos\theta$$

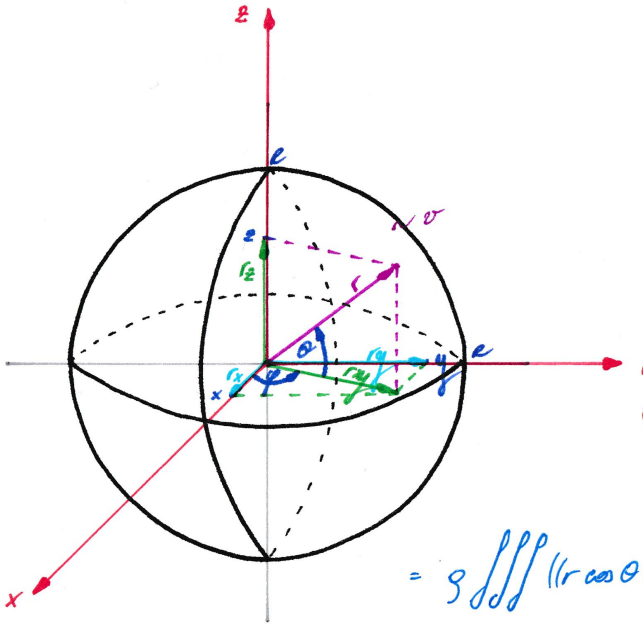
$$III)\ \int \frac{2\pi}{3}\ell^3\cos\theta\,d\theta = \frac{2\pi}{3}\ell^3\int\cos\theta\,d\theta = \frac{2\pi}{3}\ell^3\left[\sin\theta + C\right]_{-\pi/2}^{\pi/2} =$$

$$= \frac{2\pi}{3}\ell^3\left[\left(\sin\tfrac{\pi}{2} + C\right) - \left(\sin\left(-\tfrac{\pi}{2}\right) + C\right)\right] = \frac{2\pi}{3}\ell^3 \cdot 2 = \frac{4\pi}{3}\ell^3$$

AM II

$$x^2 + y^2 + z^2 = R^2$$

$$z(x,y) = \sqrt{-x^2 - y^2 + R^2}$$



$$I_z = \iiint_v r^2\, dm$$

$$\frac{r_z}{r} = \operatorname{sen}\theta \Rightarrow r_z = r\operatorname{sen}\theta$$

$$\frac{r_{xy}}{r} = \cos\theta \Rightarrow r_{xy} = r\cos\theta$$

$$\frac{x}{r_{xy}} = \cos\varphi \Rightarrow x = r\cos\theta\cos\varphi$$

$$\frac{y}{r_{xy}} = \operatorname{sen}\varphi \Rightarrow y = r\cos\theta\operatorname{sen}\varphi$$

$$\Rightarrow \begin{cases} \text{I)}\ x = r\cos\theta\cos\varphi \\ \text{II)}\ y = r\cos\theta\operatorname{sen}\varphi \\ \text{III)}\ z = r\sin\theta \end{cases}$$

$$\text{IV)}\ J = r^2\cos\theta$$

$$I_z = \iiint (x^2 + y^2)\, dm = \quad \rho = \frac{m}{v} = \frac{dm}{dv} \Rightarrow dm = \rho\, dv$$

$$dv = d\theta\, d\varphi\, dr$$

$$= \rho \iiint \left((r\cos\theta\cos\varphi)^2 + (r\cos\theta\operatorname{sen}\varphi)^2\right)\cdot r^2\cos\theta\, d\theta\, d\varphi\, dr =$$

$$= \rho \iiint r^2\cos^2\theta\,(\underbrace{\operatorname{sen}^2\varphi + \cos^2\varphi}_{=1})\cdot r^2\cos\theta\, d\theta\, d\varphi\, dr = \rho \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\pi} d\varphi \int_0^{R} r^4\cos^3\theta\, dr$$

$$v_{r\varphi\theta}: \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le 2\pi \\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{cases}$$

$$\text{I)}\ \int r^4\cos^3\theta\, dr = \cos^3\theta \int r^4\, dr = \cos^3\theta\left[\frac{1}{5}r^5 + C\right]_0^R = \frac{1}{5}R^5\cos^3\theta$$

$$\text{II)}\ \int \frac{1}{5}R^5\cos^3\theta\, d\varphi = \frac{1}{5}R^5\cos\theta \int d\varphi = \frac{1}{5}R^5\cos\theta\left[\varphi + C\right]_0^{2\pi} = \frac{2\pi}{5}R^5\cos^3\theta$$

III) $\int \frac{2\pi}{5} \ell^5 \cos^3\theta \, d\theta = \frac{2\pi}{5} \ell^5 \int \cos^3\theta \, d\theta$ IV)

IV) $\int \cos^3\theta \, d\theta = \int \cos^2\theta \cdot \cos\theta \, d\theta =$ $\sin^2\theta + \cos^2\theta = 1$, $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= \int (1 - \sin^2\theta)\cos\theta \, d\theta =$ $\sin\theta = t$, $\cos\theta \, d\theta = dt$ $=$

$= \int (1 - t^2)\, dt = \int dt - \int t^2 dt = -\frac{1}{3}t^3 + t = \left[-\frac{1}{3}\sin^3\theta + \sin\theta + C \right]_{-\pi/2}^{\pi/2} =$

$= \left[-\frac{1}{3}\sin^3(\pi/2) + \sin(\pi/2) + C \right] - \left[-\frac{1}{3}\sin^3(-\pi/2) + \sin(-\pi/2) + C \right] =$

$= -\frac{1}{3} + 1 - \frac{1}{3} + 1 = \frac{6}{3} - \frac{2}{3} = \frac{4}{3}$

$\Rightarrow \underline{I}_z = \rho \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\pi} d\varphi \int_0^{\ell} r^4 \cos^3\theta \, dr = \rho \cdot \frac{8\pi}{15}\ell^5 = \frac{3m}{4\pi\ell^3} \cdot \frac{8\pi}{15}\ell^5 = \frac{2}{5} m\ell^2$

$\Rightarrow \underline{I}_z = \frac{2}{5} m\ell^2$

e
-mechanik
.COM
.PL